Journal of Engineering Mathematics **24**: 323–341, 1990.

Traveling wave solutions of nonlinear partial differential equations

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Received 14 November 1989; accepted 22 January 1990

Abstract. Elementary transformations are utilized to obtain traveling wave solutions of some diffusion and wave equations, including long wave equations and wave equations the nonlinearity of which consists of a linear combination of periodic functions, either trigonometric or elliptic. In particular, a theorem is established relating the solutions of a single cosine equation and a double sine–cosine equation. It is shown that the latter admits a Bäcklund Transformation.

1. Introduction

This paper is concerned with finding traveling wave (TW)-solutions of some nonlinear partial differential equations, primarily in terms of the integral of a to be determined function set equal to a linear combination of the independent variables. This procedure reduces the problem to finding the solution of an ordinary differential equation of order one less than the order of the partial differential equation and, while this of course means the potential loss of some solutions, the solutions obtained often yield direct information about the structure of a solution so that more general solutions may be obtained by extrapolation. The method has been found to be quite useful in finding solutions of the Klein–Gordon (KG)-equation [1], including the special case which has become known as the Sine–Gordon (SG)-equation [2], [3].

The first equation to be considered is a nonlinear diffusion equation for which the method reduces the problem to the solution of an Abel differential equation. Although for this problem, the method is useful primarily in an inverse manner, i.e., choose a function and see for what equation it provides a TW-solution, exact functional forms are determined for some physically important problems. Further, an approximate method is derived for some polynomial approximations of the nonlinear term.

A brief discussion of nonlinear wave equations, including the KG-equation, is given before considering variations of the SG-equation to allow for the addition of forcing and damping terms. For a constant forcing term, a solution is obtained by quadrature, using an alternative direct method. This solution generalizes a previously obtained solution of the SG-equation. When damping is added to the equation, an approximate solution is obtained for a polynomial approximation (nth partial sum of the Taylor series) of the sine function.

The fifth section is concerned with long wave equations. The Korteweg–de Vries (KdV)-equation and its various generalizations have been shown to model a surprising diversity of finite-amplitude dispersive wave propagation in many physically important contexts [4]. TW-solutions of some generalized KdV-equations are obtained. Since many of these equations admit Bäcklund transformations (BT), additional solutions may be obtained from known solutions [5].

The sixth section is concerned with nonlinear wave equations the inhomogeneous term of which consists of a linear combination of periodic terms with special arguments. The usual trigonometric functions, except for the cosine, are discussed.

The seventh section is concerned with the cosine equations. The solution for an n-tuple cosine equation would provide an approximate solution for a nonlinear wave equation the inhomogeneous term of which may be approximated by a partial sum of a Fourier cosine series, a result entirely analogous to the previously obtained result for a Fourier sine series approximation [2]. Further, it is shown that each TW-solution of the single cosine equation immediately leads to a TW-solution of a double sine–cosine equation, a special case of which is a double cosine equation. It is shown that the double equation admits a BT.

The final section is concerned with inhomogeneous terms which are Jacobian elliptic functions. The solutions obtained contain the solutions for the SG-equation, the cosine equation and the tangent equation as special limiting cases.

2. Nonlinear diffusion

The first application of the method will be to nonlinear diffusion problems. The most simple nonlinear diffusion equation is

$$u_t = u_{xx} + H(u) \tag{1}$$

where x is a spatial variable, t the time, H a given nonlinear function and partial derivatives are denoted by subscripts. The procedure is to look for a solution in the form

$$\int F^{-1}(u)\,\mathrm{d}u = rx + st \equiv \zeta \tag{2}$$

where r and s are constants with r assumed always positive, so the direction of the wave is determined by the sign of s. Inclusion of r is a convenient way to allow for scaling of the variables. Substitution of equation (2) into equation (1) leads to the ordinary differential equation

$$r^2FF' - sF' + H(u) = 0 \tag{3}$$

where the prime denotes differentiation with respect to the argument. Equation (3) is classified more easily when written in terms of $F_1 = 1/F$, i.e.,

$$r^2F_1' + sF_1^2 - HF_1^3 = 0\,. \tag{4}$$

Thus, the procedure reduces the problem of finding a TW-solution of the nonlinear diffusion problem governed by equation (1) to the solution of an Abel differential equation, an equation for which not very many closed-form solutions are known. Consequently, equation (3) is applied most easily in an inverse manner, i.e., particular choices of F will yield TW-solutions of equation (1) with H determined from equation (3).

The extension to more difficult diffusion equations, e.g.,

$$u_t = u_{xx} + M(u)u_x^2 + H(u) \tag{5}$$

with an additional function M leads to a more difficult Abel equation; however, used inversely, a hierarchy of solutions may be determined by choosing F and also either M or H. Clearly, the method is applicable to any diffusion equation with coefficients and extra function which depend only on u. Also, the extension to equations dependent on additional spatial variables causes no further difficulties. In order to dispel the notion that the method merely reduces to an iterative guessing game, some exact results and an approximate method, which will prove quite useful in the sequel, will be discussed.

Many diffusion problems are governed by the equation

$$u_t = [K(u)u_x]_x \,. \tag{6}$$

An exact solution may be obtained for the appropriate ordinary differential equation so that a TW-solution is given by

$$\int [K(u)/(au + b)]\,\mathrm{d}u = s\zeta/ar^2 \tag{7}$$

where a and b are arbitrary constants.

The equation

$$\theta_{xx} - \alpha\theta_x^2 = \exp[(\alpha - \gamma)\theta]\theta_t \tag{8}$$

with constant α and γ may be transformed into the form of equation (6) by introduction of the transformation $u = \exp(-\gamma\theta)$; namely,

$$u_t = [u^{-1+\alpha/\gamma}u_x]_x \,, \tag{9}$$

so that a TW-solution is given by equation (7) with

$$K(u) = u^{-1+\alpha/\gamma}$$

The equation

$$w(u)u_t = [K(u)u_x]_x \tag{10}$$

admits a TW-solution with

$$F(u) = \left[s\int w(u)\,\mathrm{d}u\right]\Big/ r^2K(u)\,, \tag{11}$$

which is consistent with equation (7).

The equation

$$u_t = [D(u)u_x - K(u)]_x \tag{12}$$

admits a TW-solution

$$a\int \{D(u)/[a(su/r + K)/r + b]\}\,\mathrm{d}u = \zeta \,. \tag{13}$$

Physical problems for which these equations are appropriate are discussed in reference [6].

A particularly useful choice for F is a polynomial in u. If F is chosen to be a polynomial of order n, then H will be a polynomial of order $2n-1$. If $H(u)$ may be approximated by a polynomial of odd order, then a corresponding approximate solution may be obtained; an interesting example will be given in Section 4.

More difficult equations, e.g.,

$$u_t = u_{xx} + \exp(-b_1/u) \tag{14}$$

where b_1 is constant, also may be treated approximately. Thus, using an approximation for F in terms of reciprocal powers of u, gives

$$\int \left[\mathrm{d}u \Big/ \sum_{i=0}^{n} (a_i/u^i) \right] = \zeta \tag{15}$$

where the a_i's may be determined recursively when the exponential term in equation (14) is approximated by an appropriate partial sum of its series expansion in reciprocal powers of u.

3. Nonlinear wave equations

TW-solutions of various nonlinear wave equations may be obtained in an analogous fashion. The usual KG-equation may be written as

$$\phi_{\xi\eta} = v'(\phi) \tag{16}$$

in terms of two (characteristic) variables. As shown previously [1], introduction of the transformation

$$\mathrm{d}w = v^{-1/2}\, \mathrm{d}\phi\,, \tag{17}$$

i.e., $F(\phi) = v^{1/2}(\phi)$, immediately leads to a TW-solution

$$w = A\xi + 2\eta/A$$

where A is constant. For example, with a potential function

$$v(\phi) = (\phi^2 + a^2)(1-\phi^2)\nu/8(1+a^2)$$

where a and ν are constants, a TW-solution is given by

$$2(a^2+1)^{1/2}\phi(\phi^2+a^2)^{1/2}/a^2(1-\phi^2) = \sinh[\nu(A\xi + 2\eta/A)/2^{1/2}]\,. \tag{18}$$

From this structural form, additional solutions may be obtained.

A similar equation is the Dodd–Bullough equation [7]

$$\phi_{\xi\eta} = \mathrm{e}^{2\phi} - \mathrm{e}^{-\phi}$$

for which a solution is given by

$$\int z^{-1/2}(z+2^{1/3})^{-1/2}[(z-2^{-2/3})^2+3/2^{4/3}]^{-1/2}\,dz = A\xi + 2\eta/A$$

where $z = \exp\phi$. The integral may be expressed in terms of elliptic functions.

The equation

$$u_{yy} - [\psi^2(u)u_x]_x = 0$$

admits a TW-solution

$$\int [A^2\psi^2 - B^2]\,du = Ax + By\,.$$

Additional examples of the present technique may be found in previous papers [3], [8].

4. The modified Sine–Gordon equation

The particular case of the KG-equation which has become known as the SG-equation has been utilized as the model equation for many important physical phenomena. While the SG-equation has been studied using the present procedure, [1], it has been shown that an alternate approach, based on a special transformation, leads to more detailed results [2]. Thus, both approaches will be utilized.

In some cases, the SG-equation occurs with additional terms due to forcing and damping. With a constant forcing term E, the equation may be written as

$$\sigma_{\xi\eta} = \sin\sigma + E\,. \tag{19}$$

Introduction of the transformation

$$\sigma = 2\arctan N(A\xi + B\eta) \tag{20}$$

where N is a twice-continuously differentiable function and A and B are constants leads to the ordinary differential equation

$$L[N] \equiv N''/N - 2N'^2/(1+N^2) = E(1+N^2)/2ABN + 1/AB\,, \tag{21}$$

which may be integrated once to give

$$N'^2/(1+N^2) = C - 1/AB + CN^2 - [E(1+N^2)/AB]\arctan N$$

where C is an arbitrary constant. When E vanishes, this first-integral reduces to a solution obtained previously for the SG-equation [2].

The damped and forced SG-equation to be considered may be written as

$$\sigma_{\xi\eta} = \sin\sigma + E + D\sigma_\xi \tag{22}$$

where D is constant. Since the use of the transformation given by equation (2) leads to an (Abel) equation for which a first integral has not been obtained, an approximate solution may be obtained through the use of the results of Section 2. Thus, looking for a solution of equation (22) in the form

$$\int F^{-1}(\sigma)\,\mathrm{d}\sigma = A\xi + B\eta\,, \tag{23}$$

leads to

$$ABFF' - ADF = E + \sin\sigma\,. \tag{24}$$

Replacing the sin σ term by a partial sum of its series expansion (a polynomial of odd order) allows a solution in the form of a polynomial. As a specific example, chosen for algebraic simplicity, if the first two terms of the sine series are retained, the choice

$$F(\sigma) = a_0 + a_1\sigma + a_2\sigma^2$$

provides an approximate solution. Since it follows immediately that $AB < 0$, the choice $AB = -1$ will be made. The coefficients are easily determined as

$$a_0 = 3^{1/2}(-1 + 2A^2D^2/9)\,,\qquad a_1 = -AD/3\,,\qquad a_2^2 = 1/12\,,$$

and the wave speed a is determined from

$$2(-1 + 2A^2D^2/9)AD3^{1/2}/3 + E = 0\,.$$

Higher-order approximations may be obtained in a similar fashion.

The SG-equation

$$\sigma_{xx} - \sigma_{tt} = \sin\sigma \tag{25}$$

admits a similarity solution in terms of the variable $z = xt$. This solution has been utilized to discuss the propagation of ultrashort light pulses in an amplifier [10], [11]. With $\sigma(x, t) = \sigma_1(z)$, this leads to

$$z\sigma_1'' + \sigma_1' - \sin\sigma_1 = 0\,. \tag{26}$$

Introduction of a new dependent variable transforms equation (26) to a special case of the equation that defines the third Painlevé trancendent [12]; however, direct numerical treatment of equation (26) is the expedient choice. Alternatively, introduction of the variable change $z_1 = \log z$, $\sigma_1(z) = \sigma_2(z_1)$ removes the first-order derivative term and gives the differential equation

$$\sigma_2'' = \exp(z_1)\sin\sigma_2\,. \tag{27}$$

More generally, consider the equation for $\psi(z_1)$,

$$\psi'' = \Omega(z_1)\sin\psi\,. \tag{28}$$

Introduction of the transformation $\psi = 2\arctan N_1(z_1)$ leads to the ordinary differential equation

$$L[N_1] = \Omega(z_1)\,. \tag{29}$$

A first integral of equation (29) may be written as

$$(N_1')^2 = (1+N_1^2)^2\left[A - \int \Omega\,\mathrm{d}(1+N_1^2)^{-1}\right] \tag{30}$$

where A is an arbitrary constant. Equation (27), of course, corresponds to $\Omega = \exp(z_1)$.

5. Long wave equations

The model equations of Korteweg and de Vries (KdV) and Boussinesq originated as approximate governing equations for long water waves of small amplitude; however, these equations have been found to be applicable to physically interesting problems in other contexts. Since the present method is too simple to produce any new results for these extensively studied equations, the primary focus of this section is directed toward generalizations, although it is shown that special cases of the classical results may be obtained.

The starting point is the generalized KdV equation

$$u_t + Au^n u_x = Bu_{xxx}\,. \tag{31}$$

Looking for a solution in the form of equation (2) leads to the ordinary differential equation

$$G'' = s/Br^3 + Au^n/Br^2 \tag{32}$$

where $G = F^2/2$. The solution is

$$F^2 = a + bu + su^2/Br^3 + 2Au^{n+2}/(n+1)(n+2)Br^2 \tag{33}$$

where a and b are arbitrary constants. The KdV-equation corresponds to $n = 1$, and the solutions may be expressed in terms of elliptic function. Comparable results are obtained for the modified KdV (mKdV)-equation which corresponds to $n = 2$. Larger values of n yield particular solutions for the appropriate equations.

An alternative equation for long wave approximations is the Benjamin–Bona–Mahony equation

$$u_t + u_x + Auu_x = Bu_{xxt} \tag{34}$$

in which arbitrary constant coefficients have been introduced. For this equation, the value for F is obtained as

$$F^2 = a + bu + (r+s)u^2/Br^2s + Au^3/3Brs \tag{35}$$

which differs from the solution given by equation (33) (with $n = 1$) for the KdV-equation only in the coefficients of the last two terms.

A combination of the KdV- and mKdV-equations [13],

$$u_t + Auu_x + Bu^2u_x + Cu_{xxx} = 0\,,$$

leads to the solution

$$G = a + bu - su^2/2r^3C - Au^3/6r^2C - Bu^4/12r^2C\,.$$

An equation which occurs in nonlinear dispersive systems with dissipation is the KdV–Burgers equation. Since a combined mKdV–Burgers equation has been found to be of interest [13], a generalized mKdV–Burgers equation will be considered; specifically,

$$u_t + B^2u^nu_x - Au_{xx} = Cu_{xxx}\,. \tag{36}$$

Looking for a solution in the form of equation (2) leads to the differential equation

$$FF' + AF/Cr = su/Cr^3 + B^2u^{n+1}/C(n+1)r^2 + D \tag{37}$$

where D is constant. From the analysis of Section 2, a polynomial solution for F may be obtained for even n. In particularly, for $n = 2$,

$$F = \sum_{i=0}^{2} (a_iu^i)$$

with $a_0 = 3CDr/2A$, $a_1 = -A/3Cr$, $a_2 = B/r(6C)^{1/2}$, with s determined from

$$s = Cr^3(2a_0a_2 + a_1^2 + Aa_1/Cr)\,.$$

Some comments about the KdV–Burgers equation (equation (37) with $n = 1$) are appropriate. It is of interest to compare this equation with Fisher's equation [9] from population dynamics, i.e.,

$$u_t = D_1u_{xx} + A_1u + B_1u^2 \tag{38}$$

with specified coefficients A_1, B_1 and D_1. A TW-solution of equation (38) leads to the differential equation

$$FF' + sF/D_1r^2 = -A_1u/D_1r^2 - B_1u^2/Dr^2\,. \tag{39}$$

Equation (39) is equal to the corresponding equation for the KdV–Burgers equation if the integration constant D is chosen to be zero. Since extensive numerical studies of the latter have been carried out [14], these results should be applicable to Fisher's equation.

The Boussinesq equation

$$u_{tt} - u_{xx} - (3u^2)_{xx} - u_{xxxx} = 0 \tag{40}$$

yields a polynomial solution

$$G=\sum_{i=0}^{3}(a_i u^i)$$

where a_0 and a_1 are arbitrary, $a_2=-(r^2-s^2)/2r^4$, $a_3=-r^{-2}$.

There has been some interest in generalization of the KdV-equation with higher-order x-derivatives [15], [16] [17], e.g., the equation

$$u_t+Au^m u_x+Bu_{5x}=0 \tag{41}$$

where the notation indicates five derivatives with respect to x. The appropriate corresponding equation for G is

$$G^* \equiv 2GG''-\tfrac{1}{2}G'^2=-Au^{m+2}/r^4B(m+1)(m+2)-su^2/2Br^5+au+b\,. \tag{42}$$

There is a polynomial solution for $m=2$; namely

$$G=a_0+a_1u+a_3u^3 \tag{43}$$

with $a_0=a/12a_3$, $a_1=s(-10B/A)^{1/2}/6Br^3$, $a_3^2=-A/90Br^4$, which satisfies equation (42) with $b=-a_1^2/2$. Assuming $A>0$, this solution requires $B<0$ in order to be real.

The fifth-order equation [17]

$$u_t+Au^2u_x+Buu_{xxx}+Cu_xu_{xx}+Du_{5x}=0 \tag{44}$$

yields a solution

$$G=a_0+a_1u+a_3u^3$$

where a_3 is determined from

$$90a_3^2+3(C+2B/r)a_3/Dr^2+A/Dr^4=0\,,$$

$$a_1=-(s/Dr^5)/(18a_3+C/Dr^2)\,,$$

and a_0 is arbitrary.

The appropriate equation for G for the seventh-order equation

$$u_t+Au^nu_x+Bu_{7x}=0 \tag{45}$$

is

$$r^7B\{G'G^{*\prime\prime}+2GG^{*\prime\prime\prime}\}'+Aru^n+s=0$$

which may be integrated once directly. If the constant of integration is chosen to be zero, there is a simple solution for G; namely,

$$G = -s/(r^7 B 360 a_3^2) + a_3 u^3$$

where $a_3^3 = -A/(2520 B r^6)$.

An equation which includes the KdV- and SG-equations as limiting cases is [17]

$$u_{\xi\eta} + A u_\xi^2 u_{\xi\xi} + B u_{\xi\xi\xi\xi} - \alpha \sin u = 0 \,. \tag{46}$$

The corresponding equation for G is

$$G^{*\prime} + sG'/Br^3 + A(G^2)'/B = (\alpha \sin u)/Br^4 \,.$$

Integrating once, approximating $\cos u$ by the first three terms of its series expansion and looking for a polynomial solution, gives

$$G = a_0 + a_2 u^2 \,,$$

$$a_2^2 = -\alpha/(Ar^4 4!) \,, \qquad a_0 = (\alpha/2Br^4 - 2a_2^2 - sa_2/Br^3)B/2Aa_2 \,,$$

and the constant of integration is related to s by

$$C = 4a_0 a_2 + sa_0/Br^3 + Aa_0^2/B + \alpha/Br^4 \,.$$

Assuming $\alpha > 0$, the solution will be real if $A < 0$.

It is possible to treat analogous equations with additional independent variables in an entirely similar way. Looking for a solution of [18]

$$u_{xt} + (3u^2)_{xx} + u_{xxxx} + Du_{yy} = 0 \tag{47}$$

in the form

$$\int [\mathrm{d}u/F(u)] = Ax + By + Ct \,,$$

leads to a solution

$$G = a_0 + a_1 u - (AC + DB^2)u^2/2 - u^3/A^2$$

where a_0 and a_1 are arbitrary.

6. Wave equations with N-tuple periodic terms

The equations to be considered are of the form

$$u_{\xi\eta} = \sum_{i=1}^{n} [A_i T_i(u)] \tag{48}$$

where the T_i's are periodic functions with special arguments of u, either the trigonometric functions or elliptic functions. With specified coefficients A_i and $T_i = \sin(u/i)$, equation (48) is an n-tuple SG-equation of particular interest in nonlinear optics, providing the equations which govern the propagation of ultrashort light pulses when the effects of level degeneracy must be included (Q-transition). The solution of this problem was discussed in a recent paper [2] for the more general case of arbitrary coefficients A_i thus providing, in addition, an approximate solution for an arbitrary KG-equation, the inhomogeneous term of which may be approximated by a partial sum of a Fourier sine series. Solutions (by quadrature) for the other trigonometric functions will be given in this section.

The first equation to be discussed is the KG-equation

$$u_{\xi\eta} = \tan u \,. \tag{49}$$

Introduction of the transformation

$$u = 2 \arctan N(A\xi + B\eta) \tag{50}$$

leads to the differential equation [see equation (21)]

$$ABL[N] = (1 + N^2)/(1 - N^2) \,. \tag{51}$$

A first integral of equation (51) is given by

$$2ABN'^2 = (1 + N^2)^2\{\ln[(1 + N^2)/(1 - N^2)] + D\} \,, \tag{52}$$

so that the solution of equation (49) is reduced to a quadrature. D is a constant of integration.

Proceeding to the double-tangent equation

$$u_{\xi\eta} = R \tan(u/2) + S \tan u \tag{53}$$

where R and S are constants, the transformation given by equation (49) leads to the differential equation

$$2ABL[N] = (1 + N^2)[(R + 2S - RN^2)/(1 - N^2)] \,. \tag{54}$$

A first-integral of equation (54) is given by

$$2ABN'^2 = (1 + N^2)^2\{\ln[1 + N^2)^{R+S}/(1 - N^2)^S] + D\} \,. \tag{55}$$

Equation (55) contains equation (51) as the special case obtained by letting $R = 0$, $S = 1$.

The triple-tangent equation will be taken in the form

$$u_{\xi\eta} = R \tan(u/3) + S \tan(2u/3) + T \tan u \tag{56}$$

where T is constant. The appropriate transformation is

$$u = 3 \arctan N \,, \tag{57}$$

which leads to the differential equation

$$3ABL[N]/(1+N^2) = R + 2S/(1-N^2) + T(3-N^2)/(1-3N^2) . \tag{58}$$

A first integral of equation (58) is given by

$$3ABN'^2 = (1+N^2)^2\{\ln[(1+N^2)^{R+S+T}/(1-N^2)^S(1-N^2)^{2T/3}] + D\} . \tag{59}$$

With minor changes in the definition of u, solutions of equations (49) and (53) are contained as special limiting cases of equation (59). This property of the solutions of n-tuple equations containing solutions of $(n-1)$-tuple, $(n-2)$-tuple, etc. holds for each of the trigonometric equations. Clearly, solutions for four-tuple, five-tuple, etc., equations may be obtained in a similar fashion.

The transformation given by equation (50) applied to the equation

$$u_{\xi\eta} = \cot u \tag{60}$$

leads to the differential equation

$$4ABN^2L[N] = 1 - N^4 ,$$

which has the first integral

$$2ABN'^2 = (1+N^2)^2\{\ln[N/(1+N^2)] + D\} .$$

For the double-cotangent equation

$$u_{\xi\eta} = R\cot(u/2) + S\cot u ,$$

the same transformation leads to

$$4ABN^2L[N] = 2R + S - SN^2 ,$$

for which a first-integral is given by

$$2ABN'^2 = (1+N^2)^2\{\ln[N^{2R+S}/(1+N^2)^{R+S}] + D\} .$$

For the triple-cotangent equation

$$u_{\xi\eta} = R\cot(u/3) + S\cot(2u/3) + T\cot u ,$$

the transformation given by equation (37) leads to

$$3ABN^2L[N]/(1+N^2) = R + S(1-N^2)/2 + T(1-3N^2)/(3-N^2) ,$$

for which a first integral is given by

$$3ABN'^2 = (1+N^2)^2\{\ln[N^{2R+S+2T/3}(N^2-3)^{2T/3}/(1+N^2)^{R+S+T}] + D\}\,.$$

The equation

$$u_{\xi\eta} = \sec u$$

leads to the differential equation

$$2N(1-N^2)ABL[N] = (1+N^2)^2$$

after application of the transformation given by equation (50). A first integral is given by

$$ABN'^2 = (1+N^2)^2[\operatorname{artanh} N + D]\,.$$

For the double-secant equation

$$u_{\xi\eta} = R\sec(u/2) + S\sec u\,,$$

the transformation

$$u = 4\arctan N \tag{61}$$

leads to

$$4ABL[N]N/(1+N^2) = (1+N^2)(1-N^2)^{-1}[R + S(1+N^2)(1-N^2)/(1-6N^2+N^4)]\,.$$

A first integral of this equation is given by

$$4ABN'^2/(1+N^2)^2 = 2R\operatorname{artanh} N + S[(2^{1/2}-1)a_2^{-1/2}\operatorname{artanh}(Na_2^{-1/2}) - (1+2^{1/2})a_1^{-1/2}\operatorname{artanh}(Na_1^{-1/2})] + D\,.$$

For the triple-secant equation

$$u_{\xi\eta} = R\sec(u/3) + S\sec(2u/3) + T\sec u\,,$$

the appropriate transformation is

$$u = 6\arctan N\,. \tag{62}$$

This gives

$$6ABL[N]N/(1+N^2) = [(1+N^2)/(1-N^2)][R + S(1+N^2)(1-N^2)/(1-6N^2+N^4) + T(1+N^2)^2/(1-14N^2+N^4)]\,.$$

A first integral is given by

$$3ABN'^2/(1+N^2)^2 = (R - T/3)\,\text{artanh}\,N + S(2^{1/2}-1)(2a_2)^{-1/2}\,\text{artanh}(Na_2^{-1/2})$$
$$- S(1+2^{1/2})/(2a_1)^{-1/2}\,\text{artanh}(Na_1^{-1/2}) - T[(3^{1/2}-2)/3a_4^{1/2}]\,\text{artanh}(NA_4^{-1/2})$$
$$+ T[(3^{1/2}+2)/3a_3^{1/2}]\,\text{artanh}(Na_3^{-1/2}) + D\,,$$

where $a_1 = 3 + 2^{3/2}$, $a_2 = 3 - 2^{3/2}$, $a_3 = 7 + 4(3^{1/2})$, $a_4 = 7 - 4(3^{1/2})$.

Equation (50) applied to the equation

$$u_{\xi\eta} = \csc u$$

leads to the differential equation

$$4N^2ABL[N] = (1+N^2)^2\,,$$

which has the first integral

$$2ABN'^2 = (1+N^2)^2\{\ln N + D\}\,.$$

Under the transformation given by equation (61), the double-cosecant equation leads to

$$8N^2ABL[N] = (1+N^2)^2[R + S(1+N^2)/2(1-N^2)]\,,$$

which has the first integral

$$8ABN'^2 = (1+N^2)^2\{\ln[(N^2)^{R+S/2}/(1-N^2)^S] + D\}\,.$$

For the triple-cosecant equation, the appropriate transformation is equation (62). The differential equation which results is

$$12ABN^2L[N]/(1+N^2)^2 = R + S(1+N^2)/2(1-N^2) + T(1+N^2)^2/(3-10N^2+3N^4)\,.$$

A first integral is given by

$$12ABN'^2 = (1+N^2)^2\{\ln[(N^2)^{R+S/2+T/3}(1-N^2)^{-S}\{(3-N^2)/(1-3N^3)\}^{2T/3}] + D\}\,.$$

7. Cosine equations

The transformation given by equation (50) is appropriate for the equation

$$u_{\xi\eta} = \cos u\,. \tag{63}$$

This leads to the differential equation

$$2ABL[N] = 1 - N^2\,, \tag{64}$$

for which a first integral is given by

$$ABN'^2 = (1 + N^2)(D + N + DN^2) . \tag{65}$$

For the double-cosine equation

$$u_{\xi\eta} = R\cos(u/2) + S\cos u ,$$

the appropriate transformation is equation (61). The resulting differential equation is

$$4ABL[N]N = R(1 - N^2) + S(1 - 6N^2 + N^4)/(1 + N^2) ,$$

for which a first integral is given by

$$2ABN'^2 = N[(R - S)N^2 + R + S] + D(1 + N^2)^2 .$$

For the triple-cosine equation

$$u_{\xi\eta} = R\cos(u/3) + S\cos(2u/3) + T\cos u ,$$

the appropriate transformation is equation (62). The resulting differential equation is

$$6ABNL[N]/(1 + N^2) = -S + (R - 3T)(1 - N^2)/(1 + N^2) + 2S(1 - N^2)^2/(1 + N^2)^2 + 4T(1 - N^2)^3/(1 + N^2)^3 .$$

A first integral is given by

$$6ABN'^2 = 2(R - S + T)N(1 + N^2) + 4(S - 8T/3)N + (32T/3)N(1 + N^2)^{-1} + D(1 + N^2)^2 .$$

One of the interesting conclusions of the previous discussion of the n-tuple SG-equation was the result that each TW-solution of the single SG-equation immediately led to a corresponding TW-solution of the double SG-equation. Thus, a BT could be written for the double SG-equation, the "vacuum solution" of which led to a convenient representation of the single-soliton solution of the double equation [2]. An analogous theorem connecting solutions of single-cosine equations and a double-sine–cosine equation, a special case of which is a double-cosine equation, may be derived in an entirely similar manner.

The basic assumption is that N as given by equation (50) satisfies equation (63) and, therefore, equation (65). The proof is simple and constructive.

The starting point is direct differentiation of

$$u = 4\arctan\{cN[A_1\xi + B_1\eta]\} \tag{66}$$

where A_1, B_1 and c are to be determined constants. This procedure gives

$$u_{\xi\eta} = 2A_1B_1\sin(u/2)[N''/N - 2c^2N'^2/(1 + c^2N^2)]$$
$$= (2A_1B_1/AB)\sin(u/2)[(1 - N^2)/2N + 2(1 - c^2)(D + N + DN^2)/(1 + c^2N^2)]$$

or

$$u_{\xi\eta} = [A_1 B_1 (c^2+1)/ABc]\{\cos(u/2) + [(c^2-1)/(c^2+1)]\cos u + [D(1-c^2)/c(1+c^2)][2(1+c^2)\sin(u/2) + (c^2-1)\sin u]\}\,. \quad (67)$$

Thus, equation (66) is a solution of equation (67). Summarizing, we have

THEOREM. *If* $u = 2\arctan N[A\xi + B\eta]$ *is a solution of*

$$u_{\xi\eta} = \cos u\,,$$

then

$$u = 4\arctan\{cN[A_1\xi + B_1\eta]\}$$

is a solution of equation (67), *a double-sine–cosine equation.*

COROLLARY. *If D is chosen to be zero, then*

$$u = 4\arctan\{cN[A_1\xi + B_1\eta]\}$$

is a solution of the double-cosine equation

$$u_{\xi\eta} = R\cos(u/2) + S\cos u$$

with $S/R = (c^2-1)/(c^2+1)$, $A_1B_1 = RABc/(1+c^2)$.

The close relation between the cosine equation and equation (67) (or the double-cosine equation) means that a BT may be written for the double equation. The starting point is the BT for the usual SG-equation, i.e.,

$$[(u_1 - u_2)/2]_\xi = a\sin[(u_1+u_2)/2]\,,$$

$$[(u_1 + u_2)/2]_\eta = a^{-1}\sin[(u_1-u_2)/2]\,.$$

A simple translation of the dependent variable gives the following BT for the single-cosine equation

$$[(u_1 - u_2)/2]_\xi = a\cos[(u_1+u_2)/2]\,,$$
$$[(u_1 + u_2)/2]_\eta = a^{-1}\sin[(u_1-u_2)/2]\,, \quad (68)$$

which has the "vacuum solution"

$$u + \pi/2 = 4\arctan\exp(a\xi + \eta/a)\,.$$

There are various ways of writing a BT for the double equation, but the most practical is to write equation (68) in terms of

$$u = 2 \arctan N[A\xi \pm \eta / A] .$$

This gives

$$\begin{aligned} N_{1\xi}/(1 + N_1^2) - N_{2\xi}/(1 + N_2^2) &= a(1 - N_1 N_2)/[(1 + N_1^2)(1 + N_2^2)]^{1/2} , \\ N_{1\eta}/(1 + N_1^2) + N_{2\eta}/(1 + N_2^2) &= a^{-1}(N_1 - N_2)/[(1 + N_1^2)(1 + N_2^2)]^{1/2} , \end{aligned} \tag{69}$$

Equations (69) provide a BT for the double equation if N is determined from

$$u = 4 \arctan\{cN[\mu(a\xi + \eta / a)]\}$$

with constant μ. Setting $N_2 = 0$ in equations (69) gives

$$N_1 = [1 + \operatorname{csch}(a\xi + \eta / a)]/[\operatorname{csch}(a\xi + \eta / a) - 1]$$

as the "vacuum solution" of equations (69). Thus

$$u = 4 \arctan c\{[1 + \operatorname{csch} \mu(a\xi + \eta / a)]/[\operatorname{csch} \mu(a\xi + \eta / a) - 1]\}$$

is the corresponding solution of the double equation.

8. Elliptic functions

Equations with nonlinear periodic terms, the period of which may be varied, may be considered by using Jacobian elliptic functions. For an odd periodic nonlinear term, consider

$$u_{\xi\eta} = \operatorname{sn}(u, k) . \tag{70}$$

An appropriate transformation for this equation is

$$N(A\xi + B\eta) = \operatorname{tn}(u/2) \equiv \operatorname{sn}(u/2)/\operatorname{cn}(u/2) \tag{71}$$

where the modulus k will be omitted in order to simplify the notation. This procedure leads to the ordinary differential equation

$$\begin{aligned} L_1[N] &\equiv N''/N + [(k^2 - 2) + 2(k^2 - 1)N^2]N'^2/[1 + (1 - k^2)N^2](1 + N^2) \\ &= (1 + N^2)[1 + (1 - k^2)N^2]/AB[1 + 2N^2 + (1 - k^2)N^4] , \end{aligned}$$

which has the first integral

$$\begin{aligned} N'^2 = \{&(1 + N^2)[1 + (1 - k^2)N^2]/2ABk\} \\ &\times [D/2 + \ln\{[2(1 - k^2)N^2 + 2(1 - k)]/[2(1 - k^2)N^2 + 2(1 + k)]\}] \end{aligned} \tag{72}$$

where D is an arbitrary constant. Corresponding results for the SG-equation are obtained by taking the limit as k approaches zero.

Another equation with an odd-variable period term is

$$u_{\xi\eta} = \mathrm{tn}(u)\,. \tag{73}$$

Application of the transformation given by equation (71) gives

$$L_1[N] = (1 + N^2)[1 + (1 - k^2)N^2]/AB[1 + (k^2 - 1)N^4]\,,$$

which has the first integral

$$2(1 - k^2)^{1/2}ABN'^2 = (1 + N^2)[1 + (1 - k^2)N^2] \times \ln\{[1 + (1 - k^2)^{1/2}N^2]/[1 - (1 - k^2)^{1/2}N^2]\}\,. \tag{74}$$

In the limit as k approaches zero, equations (73), (74) reduce to the corresponding results given by equations (49) and (52) for the tangent equation.

Finally, an equation with an even-variable periodic function is

$$u_{\xi\eta} = \mathrm{cn}(u)\,. \tag{75}$$

Application of the transformation given by equation (71) gives

$$L_1[N] = [1 - (1 - k^2)N^4][1 + (2 - k^2)N^2 + (1 - k^2)N^4]^{1/2}/2AB[(1 + N^2)^2 - k^2N^4]N\,.$$

Setting

$$N'^2 = (1 + N^2)^2 H(N^2)\,,$$

where H is a continuously differentiable function, reduces the problem to a first-order ordinary differential equation for H. The solution is given by

$$2ABH/[1 + (1 - k^2)N^2] = \Big\{-\int [x(1 + x)\{1 + (1 - k^2)x\}]^{-1/2}\,\mathrm{d}x + 2\int (1 + x)^{1/2}x^{-1/2}[1 + (1 - k^2)x]^{-1/2}[(1 - k^2)x^2 + 2x + 1]^{-1/2}\,\mathrm{d}x\Big\}_{x=N^2}$$

where the notation indicates that x is to be replaced by N^2 after the integrations have been carried out. Results for the cosine equation, equation (63), are obtained by taking the limit as k approaches zero. It is possible to extend these results to double-sn, etc., equations, but the integrations are rather involved.

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